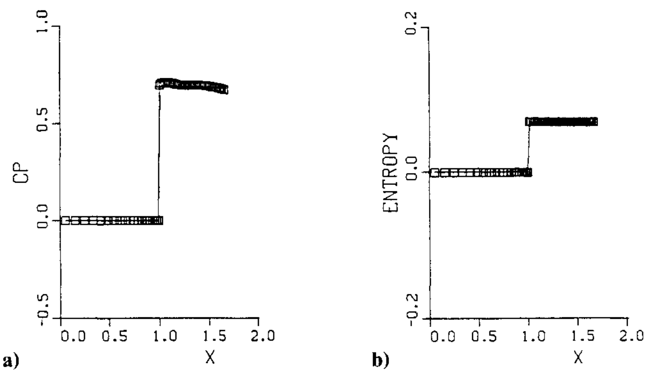

Fig. 3 Computation by the SELCO method ($M_{-\infty}=2.2$, wedge angle = 22.3 deg). Continuous lines show the exact solution for a) surface pressure coefficient and b) surface entropy.

actly. All of this is the result of the proper cell orientation toward the shock wave.

Skewing the entire grid can help to resolve accurately only one shock wave and only on the condition that the shock wave angle is known prior to the solution, so it could not be applied effectively. But if skewing of the cell can be done locally in the region of the oblique shock, it can improve the accuracy of the shock wave simulation.

The proposed SELCO method does the cell reorientation locally. The method consists of the following steps:

1) Integrate the Euler equations by the Godunov method.

2) Define the approximate shock location and the shock angle using the expressions for oblique shocks[2] (based on the fact that pressure is calculated accurately by the Godunov method).

3) Rotate the cell edges that are directed along the shock about their middle points on the shock wave angle. So, after rotation, two of the cell edges will be parallel locally to the shock wave surface.

4) Calculate the fluxes on the new edges of the cell.

5) Integrate the Euler equations in the new cell.

All these additional steps do not add much computational work, because the cell reorientation should be done only in the vicinity of the shock wave. If the shock could be resolved on one grid point, only one cell would be transformed. Our experience has shown that there is no need to extend the SELCO procedure on the cells ahead and behind the shock wave surface.

In Fig. 3 results are shown for the same flow condition as in previous cases: supersonic flow with $M_{-\infty}=2.2$ over the wedge of 22.3 deg. Calculations were done on the grid shown in Fig. 1 and in this case the SELCO method was applied. It can be concluded from the results presented in Fig. 3 that the accuracy of the shock wave modeling using the SELCO method approaches that demonstrated previously for the completely skewed grid and is superior to the accuracy of the standard Godunov method. The shock thickness in this case is minimal and much more improved compared to the simulation by the original Godunov method. Oscillations, very small in amplitude, could be observed behind the shock wave in Fig. 3. We think that they are a result of imperfect interpolation of the left- and right-side values for the Riemann problems on the upper and lower edges of the transformed cell.

Conclusions

It has been demonstrated that inaccurate modeling of the oblique shock waves produced by the Godunov method is the result of the obliqueness of the shock wave with respect to the edges of the cells of the computational grid covering the domain of integration. It is also shown that only when the shock surface is parallel to the two opposite edges of the cell can the oblique shock be accurately calculated.

A new method of local cell orientation (SELCO method) is proposed in order to allow local reorientation of the cells in the vicinity of the shock waves. The efficiency of the SELCO method is demonstrated for the simulation of the oblique shock waves in the supersonic flow using Euler equations.

Although the new SELCO method was demonstrated with the Godunov scheme, it will be effective in applications to other upwind methods that use the finite volume formulation.

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A Generalization of Caughey's Normal Mode Approach to Nonlinear Random Vibration Problems

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Introduction

THE stationary response of a stable time-invariant linear system subjected to various random excitations have been found in analytical form through complex modal analysis.[1-4] Combined with a statistical linearization technique, this method can also be applied to a nonlinear system, whether its linear part is classically damped or not. Meanwhile, no restrictions will be imposed on the excitation except the Gaussian assumption. The new method may be regarded as a generalization of Caughey's normal mode approach to nonlinear random vibration problems.[5]

Complex Modal Analysis

Consider the following multi-degree-of-freedom nonlinear system:

$$m\ddot{x}+c\dot{x}+kx+N(\dot{x},x)=w(t) \tag{1}$$

where m, c, and k are mass, damping, and stiffness matrices respectively, and $N(\dot{x},x)$ is a nonlinear function. The random excitation $w(t)$ is assumed to be Gaussian with zero mean and one of the following correlation function matrices:

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for white noise (case 1),

$$R_w(\tau) = 2\pi D\delta(\tau)$$

or for first-order filtered white noise (case 2),

$$R_w(\tau) = 2\pi D e^{-a|\tau|}$$

or for second-order filtered white noise (case 3),

$$R_w(\tau) = 2\pi D(ce^{q|\tau|} + \bar{c}e^{\bar{q}|\tau|}), \; c = \text{complex number}$$

where D is a real symmetrical non-negative matrix.

The original linear part of system (1) may be written as:

$$m\ddot{x} + c\dot{x} + kx = w(t) \tag{2}$$

When the damping is below critical, all of the eigenvalues and eigenvectors appear as complex conjugates, and the eigenvalue matrix may be written as

$$P = \begin{bmatrix} p & 0 \\ 0 & \bar{p} \end{bmatrix}, \quad p = \text{diag}[p_i], \quad i = 1,\ldots,n$$

The corresponding modal matrix is:

$$U = [Pu^T \quad u^T]^T, \quad u = [u_1 \ldots u_n \bar{u}_1 \ldots \bar{u}_n]$$

Using the complex modal transform:

$$[\dot{x}^T \quad x^T]^T = Uz$$

we can reduce Eq. (1) to a system of first-order equations with the complex modal response z:

$$\dot{z} - Pz + n(z) = f(t) \tag{3}$$

with

$$n(z) = M^{-1}u^T N(uPz, uz) = [n_i(z)]$$

$$f(t) = M^{-1}u^T w(t)$$

$$M = U^T \begin{bmatrix} 0 & m \\ m & c \end{bmatrix} U = \text{diag}[m_i]$$

Equation (3) may also be written in scalar form,

$$\dot{z}_i - p_i z_i + n_i(z) = f_i(t), \quad i = 1,2,\ldots,2n \tag{4}$$

Now we may apply the statistical linearization technique to Eq. (4). By assuming an equivalent linear system in the form:

$$\dot{z}_i - \hat{p}_i z_i = f_i(t), \quad i = 1,2,\ldots,2n \tag{5}$$

an equation error, e_i, is obtained as

$$e_i = (\hat{p}_i - p_i)z_i + n_i(z)$$

Rendering $\langle e_i \bar{e}_i \rangle$ to be a minimum, we may determine the equivalent coefficient $\hat{p}_i$ as (see Appendix)

$$\hat{p}_i = p_i - \langle n_i(z)\bar{z}_i \rangle / \langle z_i \bar{z}_i \rangle, \quad i = 1,\ldots,2n \tag{6}$$

The formula for $\langle z_i \bar{z}_j \rangle$ is[1-3]

$$\langle z_i \bar{z}_j \rangle = -g_{ij}/(p_i + \bar{p}_j) = R_{ij}, \quad \text{for case 1} \tag{7a}$$

$$\langle z_i \bar{z}_j \rangle = R_{ij}\left[\frac{1}{a - p_i} + \frac{1}{a - \bar{p}_s}\right], \quad \text{for case 2} \tag{7b}$$

$$\langle z_i \bar{z}_j \rangle = -R_{ij}\left[c\left(\frac{1}{\bar{p}_s + q} + \frac{1}{p_i + q}\right)\right.$$
$$\left.\bar{c}\left(\frac{1}{\bar{p}_s + \bar{q}} + \frac{1}{p_i + \bar{q}}\right)\right], \quad \text{for case 3} \tag{7c}$$

where g_{ij} is the element of the following matrix G:

$$G = [g_{ij}] = 2\pi M^{-1}u^T \overline{DuM}^{-T} \tag{8}$$

Since the unknown coefficients $\hat{p}_i$ in Eq. (4) are themselves functions of the responses z_i, an iteration procedure is required. Once we obtain the equivalent linear system, the correlation function matrix of the modal response, $R_z(\tau)$, can be obtained, and that of the system response x may be written as

$$R_x(\tau) = uR_z(\tau)\bar{u}^T$$

As an example, let us consider the problem of nonlinear suspension[6] which leads to the following system of equations:

$$m\ddot{x}_1 + ma\dot{x}_1 + \frac{\partial}{\partial x_1}V = F_1 \qquad I\ddot{x}_2 + Ib\dot{x}_2 + \frac{\partial}{\partial x_2}V = F_2 \tag{9a}$$

where m and I are the mass and moment of inertia of the suspended body, x_1 and x_2 the translation and rotation of the body, and F_1 and F_2 stationary Gaussian white noise excitations with zero mean. The potential nonlinearity of the suspension, V, may be approximated by

$$V = K_1x_1^2 + K_2x_2^2 + e(h_1x_1^4 + h_2x_2^4 + h_3x_1^2x_2^2)$$

where e is assumed to be a small quantity. System (9a) may be rewritten as:

$$\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + N_1(x_1,x_2) = f_1$$
$$\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 + N_2(x_1,x_2) = f_2 \tag{9b}$$

where

$$c_1 = a, \quad k_1 = 2K_1/m, \quad f_1 = F_1/m$$
$$c_2 = b, \quad k_2 = 2K_2/I \quad f_2 = F_2/I$$

and

$$N_1 = r_{11}x_1^3 + r_{12}x_1x_2^2, \quad N_2 = r_{21}x_1^2x_2 + r_{22}x_2^3$$

with

$$r_{11} = \frac{4eh_1}{m}, \quad r_{12} = \frac{2eh_3}{m}, \quad r_{21} = \frac{2eh_3}{I}, \quad r_{22} = \frac{4eh_2}{I}$$

The eigenvalue matrix of the system is

$$P = \begin{bmatrix} p & 0 \\ 0 & \bar{p} \end{bmatrix}, \quad p = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$$

where

$$p_1 = A_1 + jB_1 = -c_1/2 + j\sqrt{k_1 - c_1^2/4}$$
$$p_2 = A_2 + jB_2 = -c_2/2 + j\sqrt{k_2 - c_2^2/4}$$

The modal matrices with order 2×4 and 4×4 are, respectively:

$$u = [I_2 \quad I_2], \quad U = \begin{bmatrix} p & \bar{p} \\ I_2 & I_2 \end{bmatrix}$$

By the complex modal transform,

$$[\dot{x}_1\ \dot{x}_2\ x_1\ x_2]^T = Uz = U[z_1\ z_2\ z_3\ z_4]^T$$

Equation (9b) can be reduced to

$$\dot{z} - Pz + N(z) = M^{-1}u^T f$$

where

$$N(z) = M^{-1}u^T[N_1\ N_2]^T = [n_1(z)\ n_2(z)\ n_3(z)\ n_4(z)]^T$$

$$M^{-1} = -\left(\frac{j}{2}\right)\begin{bmatrix} B^{-1} & 0 \\ 0 & -B^{-1}\end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1/B_1 & 0 \\ 0 & 1/B_2\end{bmatrix}$$

After some elementary operations, we have

$$-B_1\langle n_1(z)\bar{z}_1\rangle =$$

$$\mathrm{Im}\langle z_1\bar{z}_3\rangle[3r_{11}\mathrm{Re}\langle z_1\bar{z}_1 + z_1\bar{z}_3\rangle + r_{12}\mathrm{Re}\langle z_2\bar{z}_2 + z_2\bar{z}_4\rangle]$$

$$+2r_{12}\mathrm{Im}\langle z_1\bar{z}_4 + z_1\bar{z}_2\rangle\mathrm{Re}\langle z_1\bar{z}_4 + z_1\bar{z}_2\rangle$$

$$+j[3R_{11}(\mathrm{Re}\langle z_1\bar{z}_1 + z_1\bar{z}_3\rangle)^2 + 2r_{12}(\mathrm{Re}\langle z_1\bar{z}_2 + z_1\bar{z}_4\rangle)^2$$

$$+r_{12}\mathrm{Re}\langle z_1\bar{z}_1 + z_1\bar{z}_3\rangle\mathrm{Re}\langle z_2\bar{z}_2 + z_2\bar{z}_4\rangle]$$

$$-B_2\langle n_2(z)\bar{z}_2\rangle =$$

$$\mathrm{Im}\langle z_2\bar{z}_4\rangle[3r_{22}\mathrm{Re}\langle z_2\bar{z}_2 + z_2\bar{z}_4\rangle + r_{21}\mathrm{Re}\langle z_1\bar{z}_1 + z_1\bar{z}_3\rangle]$$

$$+2r_{21}\mathrm{Im}\langle z_1\bar{z}_4 - z_1\bar{z}_2\rangle\mathrm{Re}\langle z_1\bar{z}_4 + z_1\bar{z}_2\rangle$$

$$+j[3r_{22}(\mathrm{re}\langle z_2\bar{z}_2 + z_2]z_4\rangle)^2 + 2r_{21}(\mathrm{Re}\langle z_1\bar{z}_4 + z_1\bar{z}_2\rangle)^2$$

$$+r_{21}\mathrm{Re}\langle z_1\bar{z}_1 + z_1\bar{z}_3\rangle\mathrm{Re}\langle z_2\bar{z}_4 + z_2\bar{z}_2\rangle]$$

According to Eq. (8) we have

$$G = \frac{1}{2}\begin{bmatrix} B^{-1}DB^{-1} & -B^{-1}DB^{-1} \\ -B^{-1}DB^{-1} & B^{-1}DB^{-1}\end{bmatrix}$$

From Eq. (7a), we have

$$\mathrm{Re}\langle z_1\bar{z}_1\rangle = -\frac{D_{11}}{4A_1B_1^2}$$

$$\mathrm{Re}\langle z_1\bar{z}_2\rangle = -\frac{D_{12}d_1}{2B_1B_2d_4}, \quad \mathrm{Im}\langle z_1\bar{z}_2\rangle = -\frac{D_{12}d_2}{2B_1B_2d_4}$$

$$\mathrm{Re}\langle z_1\bar{z}_3\rangle = \frac{D_{11}A_1}{4B_1^2d_6}, \quad \mathrm{Im}\langle z_1\bar{z}_3\rangle = -\frac{D_{11}}{4B_1d_6}$$

$$\mathrm{Re}\langle z_1\bar{z}_4\rangle = \frac{D_{12}d_1}{2B_1B_2d_5}, \quad \mathrm{Im}\langle z_1\bar{z}_4\rangle = -\frac{D_{12}d_3}{2B_1B_2d_5}$$

$$\mathrm{Re}\langle z_2\bar{z}_2\rangle = -\frac{D_{22}}{4A_2B_2^2}$$

$$\mathrm{Re}\langle z_2\bar{z}_4\rangle = \frac{D_{22}A_2}{4B_2^2d_7}, \quad \mathrm{Im}\langle z_2\bar{z}_4\rangle = -\frac{D_{22}}{4B_2d_7}$$

where

$$d_1 = A_1 + A_2, \quad d_2 = B_1 - B_2, \quad d_3 = B_1 + B_2$$

$$d_4 = d_1^2 + d_2^2, \quad d_5 = d_1^2 + d_3^2, \quad d_6 = A_1^2 + B_1^2, \quad d_7 = A_2^2 + B_2^2$$

Table 1 Numerical results for case 1

	RMA	CMA	OLP
Iteration cycles	14	18	1
$\hat{c}_1$	0.989	1.1677	1
$\hat{k}_1$	68.838	65.675	49.25
$\hat{c}_2$	2.003	2.032	2
$\hat{k}_2$	104.621	104.468	101
$\langle x_1^2\rangle$	0.2936	0.2609	0.4061
$\langle x_2^2\rangle$	0.1909	0.1885	0.1980

Table 2 Numerical results for case 2

	RMA	CMA	OLP
Iteration cycles	17	27	1
$\hat{c}_1$	0.992	1.171	1
$\hat{k}_1$	68.402	66.192	49.25
$\hat{c}_2$	2.003	2.030	2
$\hat{k}_2$	104.218	104.356	101
$\langle x_1^2\rangle$	0.2908	0.2592	0.4488
$\langle x_2^2\rangle$	0.1641	0.1618	0.1720

According to Eq. (9), we have (see Appendix)

$$a_i = -\mathrm{Re}\langle n_i\bar{z}_i\rangle/\langle z_i\bar{z}_i\rangle \qquad i = 1,2$$

$$b_i = -\mathrm{Im}\langle n_i\bar{z}_i\rangle/\langle z_i\bar{z}_i\rangle$$

From these equations we can obtain a_i and b_i through iteration. For

$$c_1 = 1, \ c_2 = 2, \ k_1 = 49.25, \ k_2 = 101,$$

$$r_{11} = 20, \ r_{12} = 10, \ r_{21} = 2.5, \ r_{22} = 5,$$

$$D_{11} = 20, \ D_{12} = 10, \text{ and } D_{22} = 40$$

both the real modal analysis (RMA)[7] and complex modal analysis (CMA) are used to solve this problem. The response of the original linear part (OLP) is taken simply for reference. The numerical results, with an iteration accuracy of order 10^{-5}, are shown in Table 1.

For filtered white noise excitation, we can use Eqs. (7b) and (7c) to calculate the covariances and obtain the solution. For example, consider the above example under first-order filtered white noise excitation. For

$$c_1 = 1, \ c_2 = 2, \ k_1 = 49.25, \ k_2 = 101,$$

$$r_{11} = 20, \ r_{12} = 10, \ r_{21} = 2.5, \ r_{22} = 5,$$

$$D_{11} = 160, \ D_{12} = 80, \ D_{22} = 320, \text{ and } a = 10$$

the numerical results, with the iteration accuracy of order 10^{-5}, are shown in Table 2.

By comparing the results for typical examples, it is shown[8,9] that all of the statistical linearization approaches, including Caughey's, Iwan and Yang's, and Atalik's, have comparable accuracies, which appear to be well within the requirements of practical engineering analysis. So does the generalized normal mode approach. The difference between the results of the real and complex analyses stems from the different minimum error criteria used in the two alternative analyses.

Conclusion

By using the complex modal analysis,[1-4] Caughey's normal mode approach to nonlinear random vibration problems has been successfully generalized in two respects. First, the random excitation may be white or colored noise, and no restrictions will be imposed on the correlations among the random excitations. Second, the linear part of the nonlinear system may be an arbitrary linear damped system. The generalization expands the usefulness of the normal mode approach and retains the previous computational advantages. The generalization does not address the limitation on the nonlinearity per se, which remains the same as in Caughey's approach.

Appendix

By using an equivalent linear element in place of the system nonlinearity, an equation error, e_i, is obtained as

$$e_i = (\hat{p}_i - p_i)z_i + n_i(z)$$

Letting

$$\hat{p}_i - p_i = a + jb$$

we have

$$\langle e_i\bar{e}_i\rangle = (a^2+b^2)\langle z_i\bar{z}_i\rangle + 2a\mathrm{Re}\langle n_i\bar{z}_i\rangle + 2b\mathrm{Im}\langle n_i\bar{z}_i\rangle + \langle n_i\bar{n}_i\rangle$$

The parameters a and b may be determined by rendering the mean square value $\langle e_i\bar{e}_i\rangle$ to be a minimum, i.e.,

$$\langle e_i\bar{e}_i\rangle = \min \tag{A1}$$

The necessary (and also sufficient in this case) conditions for Eq. (A1) to be true are

$$(\partial/\partial a)\langle e_i\bar{e}_i\rangle = 0, \quad (\partial/\partial b)\langle e_i\bar{e}_i\rangle = 0$$

These conditions lead to

$$a\langle z_i\bar{z}_i\rangle + \mathrm{Re}\langle n_i\bar{z}_i\rangle = 0$$

$$b\langle z_i\bar{z}_i\rangle + \mathrm{Im}\langle n_i\bar{z}_i\rangle = 0 \tag{A2}$$

Combining the last two equations into a single complex one, we obtain

$$(\hat{p}_i - p_i)\langle z_i\bar{z}_i\rangle + \langle n_i\bar{z}_i\rangle = 0$$

Hence

$$\hat{p}_i = p_i - \langle n_i\bar{z}_i\rangle / \langle z_i\bar{z}_i\rangle$$

Because

$$(\partial^2/\partial a^2)\langle e_i\bar{e}_i\rangle = 2\langle z_i\bar{z}_i\rangle > 0$$

$$(\partial^2/\partial b^2)\langle e_i\bar{e}_i\rangle = 2\langle n_i\bar{n}_i\rangle > 0$$

$$(\partial^2/\partial a\partial b)\langle e_i\bar{e}_i\rangle = 0$$

$\hat{p}_i$ makes $\langle e_i\bar{e}_i\rangle$ a minimum.

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Buckling of Irregular Plates by Splined Finite Strips

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Introduction

THIS Note presents the results of an investigation into an alternative finite strip method developed using cubic x-spline functions. The buckling of constant-thickness plates with rectangular shapes has been investigated by many authors. The assumptions made are those of the classical plate theory as described by Timoshenko and Gere.[1] The stability of plates using the finite element method was developed by Kapur and Hartz,[2] Anderson et al.,[3] and others. The finite strip method, with polynomials in the x direction and continuous differentiable smooth (trigonometric) series in the y direction,[4,5] is a powerful method for problems defined in rectangular domains, including plates with stiffeners. The modified finite strip method was developed for the flexural analysis of irregular plates, using spline functions instead of trigonometric series.[6] With this formulation, rectangular, nonrectangular, irregular-shaped, two-dimensional problems can be investigated. Buckling analysis of nonrectangular plates, such as trapezoidal and polygonal plates, is presented.

Theory of Buckling

The finite element theory of elastic stability of plates has been developed by Anderson et al.[3] The algebraic equation of the stability problem of a plate is as follows:

$$([K] + \lambda[K^G])\{\delta\} = \{0\} \tag{1}$$

This is an eigenvalue problem similar to the vibration problem.[7] It is equivalent to the determination of the eigenvalue of the following determinant as indicated by Cheung et al.[8]

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